MATH 5061 Problem Set 4[1]
Due date: Apr 1, 2020

Problems: (Please either type up your assignment or scan a copy of your written assignment into ONE PDF file and send it to me by email on/before the due date. Please remember to write down your name and SID. Questions marked with a † is optional.)

1. Consider the space $\mathcal{T}$ of $(0,4)$-tensors on a manifold M^m which satisfy the symmetries of the Riemann curvature tensor: for all vector fields X, Y, Z, W on M,

   - $T(X,Y,Z,W) = -T(Y,X,Z,W) = -T(X,Y,W,Z)$
   - $T(X,Y,Z,W) + T(X,Z,W,Y) + T(X,W,Y,Z) = 0$
   - $T(X,Y,Z,W) = T(Z,W,X,Y)$

   (Recall that in fact the third is implied by the other two.) Let g be a Riemannian metric on M.

   (a) If $f : M \to \mathbb{R}$ is a smooth function, then we can define a $(0,4)$-tensor $\mathbf{f}$ by
   $$\mathbf{f}(X,Y,Z,W) := \frac{f}{m(m-1)}\Big(g(X,Z)g(Y,W) - g(X,W)g(Y,Z)\Big).$$
   Show that $\mathbf{f} \in \mathcal{T}$ and that $f = \sum_{i,j=1}^m \mathbf{f}(e_i, e_j, e_i, e_j)$ where $e_1, \cdots, e_m$ is a local orthonormal basis (w.r.t. g) of vector fields on M.

   (b) If S is a trace-free symmetric $(0,2)$-tensor on M, and $m > 2$, then we can define a $(0,4)$-tensor $\mathbf{S}$ by
   $$\mathbf{S}(X,Y,Z,W) := \frac{1}{m-2}\Big(S(X,Z)g(Y,W) - S(X,W)g(Y,Z) + g(X,Z)S(Y,W) - g(X,W)S(Y,Z)\Big).$$
   Show that $\mathbf{S} \in \mathcal{T}$ and that $S(X,Y) = \sum_{i=1}^m \mathbf{S}(X, e_i, Y, e_i)$ where $e_1, \cdots, e_m$ is a local orthonormal basis (w.r.t. g) of vector fields on M.

   (c) Use (a) and (b) to show that the Riemann curvature tensor **Riem** can be written as a sum of three terms $\mathbf{Riem} = \mathbf{W} + \mathbf{Ric_0} + \mathbf{R}$ where each term is in $\mathcal{T}$, and Ric_0 denotes the trace-free part of the Ricci tensor, and R denotes the scalar curvature function. The tensor $\mathbf{W}$ is called the Weyl curvature tensor. Show that $\mathbf{W}$ has the property that its traces are zero.

2. Let (M, g) be a Riemannian manifold, and let $u \in C^\infty(M)$. Consider the conformal metric $\hat{g} = e^{2u} g$.

   (a) Calculate the Ricci and scalar curvature of $\hat{g}$ in terms of the curvature of g and covariant derivatives of u with respect to g.

   (b) Show that the Weyl curvature of $\hat{g}$ is given by $\mathbf{W}(\hat{g}) = e^{2u}\mathbf{W}(g)$.

3. (a) Show that the Weyl curvature always vanish in dimension 3. Prove that an Einstein metric on a three manifold necessarily has constant sectional curvature.

   (b) Show that the standard product metric on $S^2 \times S^2$ is Einstein, but does not have constant sectional curvature.

4. Given an affine connection D on a manifold M and a smooth function $f : M \to \mathbb{R}$, we may define the hessian of f, denoted by Hf, such that
   $$Hf(X,Y) := X(Y(f)) - (D_X Y)(f)$$
   for any vector fields X, Y on M.

   (a) Show that Hf is a $(0,2)$-tensor. Prove that Hf is symmetric for *all* $f \in C^\infty(M)$ if and only if D is torsion free.

[1]Last revised on March 16, 2020

(b) Write the local expression for Hf in a coordinate chart, i.e. compute $Hf(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$ in terms of the partial derivatives of f and the Christoffel symbols in these coordinates.

5. The Laplace-Beltrami operator Δ on a Riemannian manifold (M, g) is defined on functions $f \in C^\infty(M)$ by

$$\Delta f := \mathrm{Tr}_g(Hf)$$

where Hf is the hessian of f as defined in Question 4 with respect to the Levi-Civita connection D for (M, g). An *eigenfunction* is a function $f \in C^\infty(M)$ satisfying $\Delta f = \lambda f$ for some constant $\lambda \in \mathbb{R}$. If $\Delta f \equiv 0$, then f is said to be a harmonic function.

(a) Show that in any local coordinates $x^1, \cdots, x^m$ of M, we have (using Einstein's summation convection)

$$\Delta f = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial f}{\partial x^j} \right) = g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right)$$

where $g = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$.

(b) Show that for any real number p, the functions $(x^m)^p$ are eigenfunctions on the hyperbolic space $\mathbb{H}^m$ using the upper half-space model, i.e.

$$\mathbb{H}^m := \{(x^1, \cdots, x^m) \in \mathbb{R}^m \ : \ x^m > 0\} \text{ with the metric } g = \frac{\sum_{i=1}^m (dx^i)^2}{(x^m)^2}.$$

(c) Suppose $m = 2$. Show that a function on $\mathbb{H}^2$ is harmonic with respect to the hyperbolic metric if and only if it is harmonic for the flat Euclidean metric.